O71. Proposed by Dorin Andrica, Babes-Bolyai University, Romania.

Let n be positive integer. Prove that

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = \frac{2}{3}(n^2-1)$$

Solution by Arkady Alt , San Jose ,California, USA.

First note, that for any polynomial $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$ $(a_0 \neq 0)$ with non-zero roots $x_1, x_2, \ldots, x_n$ holds

(1) $$\sum_{i=1}^{n} \frac{1}{x_i} = -\frac{P'(0)}{P(0)}.$$

Indeed, since $P(x) = a_0(x-x_1)(x-x_2)\ldots(x-x_n)$ then

$$\sum_{i=1}^{n} \frac{1}{x-x_i} = \sum_{i=1}^{n}(\ln(x-x_i))' = \left(\sum_{i=1}^{n} \ln(x-x_i)\right)' = \left(\ln \frac{P(x)}{a_0}\right)' = \frac{P'(x)}{P(x)}.$$

Hence, $\sum_{i=1}^{n} \frac{1}{x_i} = -\frac{P'(0)}{P(0)}$.

Let $U_n(x) := \frac{T'_{n+1}(x)}{n+1} = \frac{\sin(n+1)\varphi}{\sin\varphi}$ is Chebishev Polynomial of the Second Kind.

Then $U_n(x)$ satisfies to recurrence

$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), n \in \mathbb{N}$ and $U_0(x) = 1, U_1(x) = 2x$.

Since $\frac{\sin n\varphi}{\sin\varphi} = 0 \Leftrightarrow \varphi = \frac{k\pi}{n}, n \in \mathbb{Z}$ then $U_{n-1}(x) = 0 \Leftrightarrow x = \cos\frac{k\pi}{n}, k = 1, 2, \ldots, n-1$

and $U_{n-1}(x) = 2^{n-1}\left(x - \cos\frac{\pi}{n}\right)\left(x - \cos\frac{2\pi}{n}\right)\ldots\left(x - \cos\frac{(n-1)\pi}{n}\right)$.

(If α_n is coefficient of x^n in $U_n(x)$ then α_n satisfies to recurrence $\alpha_{n+1} = 2\alpha_n, \alpha_0 = 1$).

In particularly $U_{2n-1}(x) = 2^{2n-1}\prod_{k=1}^{2n-1}\left(x - \cos\frac{k\pi}{2n}\right) =$

$$2^{2n-1}\left(x - \cos\frac{n\pi}{2n}\right)\prod_{k=1}^{n-1}\left(x - \cos\frac{k\pi}{2n}\right)\prod_{k=1}^{n-1}\left(x - \cos\frac{(2n-k)\pi}{2n}\right) =$$

$$2^{2n-1}x\prod_{k=1}^{n-1}\left(x^2 - \cos^2\frac{k\pi}{2n}\right).$$

Let $P_n(x) := \frac{U_{2n-1}(\sqrt{x})}{2\sqrt{x}}$ then $P_n(x) = 4^{n-1}\prod_{k=1}^{n-1}\left(x - \cos^2\frac{k\pi}{2n}\right)$.

Note that $U_{2n-1}(x)$ can be immediate defined by recurrence

$U_{2n+1}(x) = 2(2x^2-1)U_{2n-1}(x) - U_{2n-3}(x), n \in \mathbb{N}$ with $U_{-1}(x) = 0, U_1(x) = 2x$.

Since $U_{2n-1}(x)$ divisible by $2x$ then polynomial $P_n(x)$ satisfy to the recurrence

(2) $P_{n+1}(x) = 2(2x-1)P_n(x) - P_{n-1}(x), n \in \mathbb{N}$ with $P_0(x) = 0, P_1(x) = 1$.

Thus, appllying (1) to polynomial $P_n(x)$ we have

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2\frac{k\pi}{2n}} = -\frac{P'_n(0)}{P_n(0)}.$$

In particularly, from (2) follows recurrence

(3) $P_{n+1}(0) + 2P_n(0) + P_{n-1}(0) = 0, n \in \mathbb{N}$ with $P_0(0) = 0, P_1(0) = 1$.

Let $b_n := \dfrac{P_n(0)}{(-1)^n}$ then (3) can be rewritten as

(4) $b_{n+1} - 2b_n + b_{n-1} = 0, n \in \mathbb{N}$ with $b_0 = 0, b_1 = -1.$

Since $b_{n+1} - b_n = b_n - b_{n-1}, n \in \mathbb{N}$ then $b_n - b_{n-1} = -1, n \in \mathbb{N}$ and, therefore,

$\sum_{k=1}^{n}(b_k - b_{k-1}) = -n \Leftrightarrow b_n - b_0 = -n \Leftrightarrow b_n = -n.$

From the other hand,

$P'_{n+1}(x) = 2(2x-1)P'_n(x) + 4P_n(x) - P'_{n-1}(x), n \in \mathbb{N}$ with $P'_0(x) = 0, P'_1(x) = 0,$ then

(5) $P'_{n+1}(0) + 2P'_n(0) + P'_{n-1}(0) = 4P_n(0), n \in \mathbb{N}$ with $P'_0(0) = 0, P'_1(0) = 0.$

Let $a_n := \dfrac{P'_n(0)}{(-1)^n}$ then $\dfrac{P_n(0)}{(-1)^{n+1}} = -b_n = n$ and (5) can be rewritten as

(6) $a_{n+1} - 2a_n + a_{n-1} = 4n, n \in \mathbb{N}$ with $a_0 = a_1 = 0.$

Since sequence $\left(\dfrac{2n(n^2-1)}{3}\right)$ is particular solution of nonhomogeneous recurrence (6)

then $a_n = \dfrac{2n(n^2-1)}{3} + \alpha n + \beta$ where $\alpha = \beta = 0$ because $a_0 = a_1 = 0.$

Thus $a_n = \dfrac{2n(n^2-1)}{3}, n \in \mathbb{N}$ and

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = -\frac{P'_n(0)}{P_n(0)} = \frac{\frac{P'_n(0)}{(-1)^n}}{\frac{P_n(0)}{(-1)^{n+1}}} = \frac{a_n}{n} = \frac{2(n^2-1)}{3}.$$